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A NEW THEOREM IN THE GEOMETRY OF CONICS.[*]

By ALAN SPENCER HAWKESWORTH.

If two or more triangles, to which any conic is in common escribed, have each a vertex upon one and the same axis of said conic, and if, in each case, a circle be described, passing through the other two vertices of the said triangle, and with its center upon the said axis:

Then will the resultant system of circles have the other axis of the conic for their common radical axis, and will pass in common through the two foci, real or imaginary, as the case may be, lying upon that axis, while their "limiting points" upon the first and given axis, and line of centers of the said system of coaxal circles, will be, in turn, the imaginary or real foci of the conic.

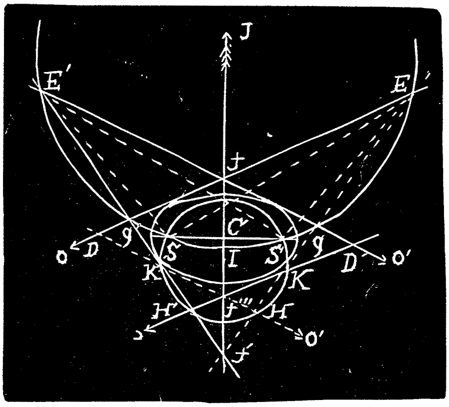

Fig. 1.

(A) For, first, take any central conic (Fig. 1), and let Et, Et' be any two tangents to it, concurring in E, and cutting its minor axis in points t and t', respectively. Draw $E't$ and $E't'$, two other tangents, symmetric about the minor axis with Et and Et', and thus concurring with them in t, t', respectively upon the axis. And let Egt' and $E'tg$ cut in g; and $E'g't'$ and Etg' cut in g', forming a pair of congruent triangles Etg and $E'tg'$, to which the conic is escribed; with a common apex t upon that conic's minor axis. Join S and S' the conic's two foci, to points E and E', in lines SE, SE', $S'E$ and $S'E'$.

Then, by a well known theorem, and by symmetry,

$$\text{angles } SEg = S'E'g' = S'Eg' = SE'g.$$

So that points $EgSS'g'E'$ are concyclic about a center lying on tt' their axis of symmetry, the conic's minor axis.

And in like manner, for any other two pairs of tangents, symmetric about that minor axis. Say, for example, $t'HK$ and $t'H'K'$; $Ht''K'$ and $H't''K$; concurring upon the minor axis in $t't''$, and cutting in KK'. Whence as before,

$$\text{angles } SHK = S'H'K' = S'HK' = SH'K,$$

and thus $HKSS'K'H'$ are concyclic about a center lying on the minor axis.

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[*] Read before the American Mathematical Society, September, 1909.

So that, conversely, if from any point t upon the minor axis of a conic, two tangents Etg' and $E'tg$ be drawn, and these be cut in E, g, respectively, by any third tangent to the curve, forming thus a triangle, with a vertex t upon the minor axis, to which the said conic is escribed, then the circle, whose center is on the minor axis, and which passes through the other two vertices E, g of said triangle, will also pass through the conic's two foci S, S'.

And hence, if two or more such triangles be drawn, as for example, Etg and $Kt''H$, then all such circles $EgSS'$ and $HKSS'$, passing in common through the two foci, have the conic's major axis for their common radical axis. While C, the conic's center, will be the position both of its imaginary foci, and of the now imaginary common "limiting points" of the said system of circles.

(B) Secondly, taking now any conic, ellipse, hyperbola, or parabola (Fig. 2), again let us have any two tangents ET and ET', concurring in E, but now cutting the conic's major axis in points T and T', respectively. Once more let us draw two fresh tangents $E'T$ and $E'T'$, but symmetric now about the major axis to ET and ET', and thus concurring with them upon it in T and T', respectively. And let them cut ETg' and $E'g'T'$ in g', and EgT' and $E'Tg$ in g; thus forming a symmetric pair of congruent triangles ETg and $E'Tg'$, to which the conic is escribed, each having a common vertex T upon that conic's major axis.

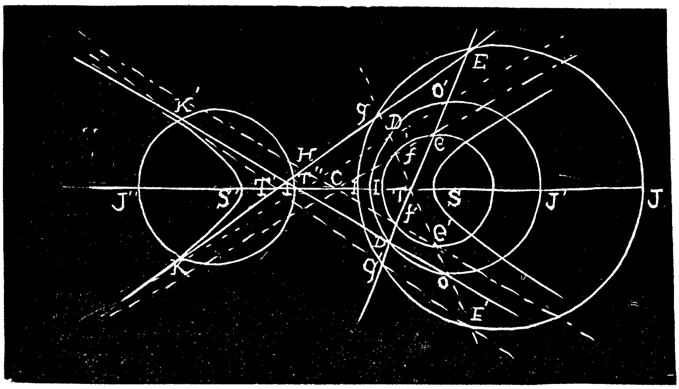


Fig. 2.

Let I and J be the two common excenters of the said two congruent triangles upon the conic's major axis, their axis of symmetry. Then IgJ, $Ig'J$, IEJ, and $IE'J$ being right angles, points I, g, E, J, E', g' are concyclic about the medial point of IJ upon the said major axis. And if the conic's foci S, S' be joined to, say E, in lines SE, $S'E'$, then, as before, $SEg'=S'Eg$; so that $SEI=S'EI$, and $SI:S'I=SE:S'E=SJ:S'J$ (Euclid VI, 3 and A).

Conversely, therefore, the circle through Eg, whose center is on TT', the conic's major axis, ever harmonically divides the inter-focal distance SS' in points I and J. Or $CI:CS=CS:CJ$, whatever may be the triangle ETg, to which the conic is escribed, and which has its vertex T upon that conic's major axis.

Thus, for example, if $KT'H'$ (Fig. 2) be such a triangle, and the circle through KH', whose center is on the major axis, cuts that axis in points I'', J'', then, as before, $CI'':CS'=CS':CJ''$.

From this it follows that the given conic's minor axis is the common radical axis of all such circles, and their common "limiting points" are the two foci of the conic. While the imaginary foci, lying at that conic's center, are now also the imaginary common intersections of that minor axis by any and all the circles.

Corollary 1. Hence when the conic is a circle, *CS* being now infinitesimal, all such circles must pass through its center. This fact is also evident from the equality of the angles *CET* and *CEg*; so that *C* is ever the common excenter of all the triangles escribed to the circle which has it for its center.

Corollary 2. On the other hand, with a parabola, any and all such circles must be concentric about the focus *S*, a fact which can also be shown as follows: The external angle *ETg* of the isosceles triangle *ETE'* is double either of the two equal interior angles *TEE'* or *TE'E*. And hence the center of the circle through *EE'g'g* must lie concyclic to the points *E*, *T*, *g* (Euclid III, 20), while, by symmetry, it must also lie on axis *TS*. And next, by a well known theorem, the circumcircle of the circumscribing triangle *ETg* must pass through *S*, the focus of the parabola. From which it follows that the center of the circle through *EE'g'g* must be either *T* or *S*. Now it cannot be *T*, since *Tg* and *TE* being equally inclined to the axis, the tangent *EgT'*, by the nature of the parabola, must be so inclined that it cuts *TE* at a greater distance from that axis and from *T* than it cuts *Tg*. Hence *S* must be the center desired. And in like manner it can be shown that *S* is also the center of the circles through *KH'HK'*, and through *ODD'O'*, etc.

Corollary 3. So that were any circumscribing triangle *DEH* to an unknown conic given, and any transversal *tt't''* or *TT'T''* of that triangle posited as the position of an unspecified axis of unknown magnitude of the said unspecified conic, then there is ever an unique solution (with the exception yet to be noted in Corollary 4, following) of one, and one conic only, which shall be inscribed or escribed, as the case may be, to the given triangle, and have the given transversal for an axis, major, or minor, as the case may be.

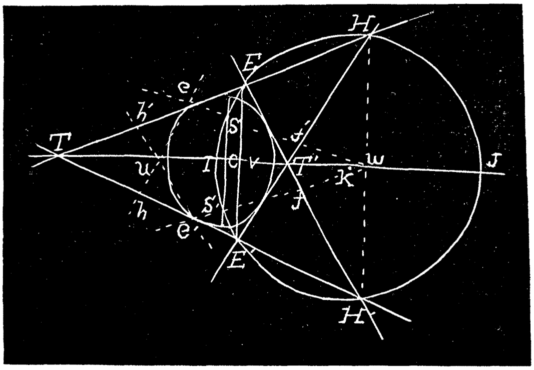
For describing a second circumscribing and congruent triangle *D'E'H'*, symmetric about the given axis, and cutting the first triangle in points *t*, *t'*, *t''*; or *T*, *T'*, *T''*; *g*, *g'*; *K*, *K'*; and *O*, *O'*, we obtain three pairs of triangles *ETg* and *E'Tg'*, *HT'K* and *H'T'K'*, *DTO* and *D'TO'*, each with a vertex upon the given axis. Wherefore, by the foregoing theorem, the circles through *Egg'E'*, *HKK'H'*, and *DOO'D'* must determine the other axis; cutting, and passing in common through the foci, when the given axis happens to be the conic's minor, but none cutting, when the given axis happens to be a major, in which latter case the external radical axis, and its two real "limiting points" give, as stated, the minor axis and foci desired.

And the foci being thus found, a perpendicular from either upon any one of the six known tangents will give us, by an elementary theorem, a point on the conic's auxiliary circle, and hence the magnitude of the major axis. So that the desired conic can now be readily drawn.

Corollary 4. But an exception to the foregoing Corollary 3 obtains when the said transversal happens to pass through a vertex of the given circumscribing triangle.

For, if it pass through a vertex, without also bisecting the angle, then the corresponding two sides of the resultant symmetric triangle form in the said vertex four concurrent rays. And no conic is possible, other than in the degenerated form of a right line, coinciding with the given axis.

While if the given axis bisects, internally, or externally, an angle of the given circumscribing triangle, then the corresponding two sides of the symmetric triangle coalesce with the two sides of the given triangle, and a symmetric quadrilateral $HE'H'ET'T$ is formed (Fig. 3). An identical figure being given, whether the said axis bisects internally the common vertical angle HTH' of the two symmetric and congruent triangles ETH and $E'TH'$, or bisects externally in common the vertical angles $E'T'H$ and $ET'H'$ of the two symmetric and congruent triangles $E'T'H$ and $ET'H'$.



Hence the conic is as yet entirely indeterminate, without further data. For we now have but one pair of symmetric and congruent triangles $E'T'H$ and $ET'H'$, having a common vertex T' upon the given axis, and thus but one circle through the points E', H, J, H', E, I, upon which the desired conic's foci, real, or imaginary, must lie. And innumerable conics are thus possible, having the given axis for either a major or a minor.

But if, in addition, we are given the position of C, the center of the required conic; then, if C falls within the limits I, J, the two excenters upon TT' of $E'T'H$ and $ET'H'$, a perpendicular through it will cut the circle through $IE'HJH'E$ in S, S', the desired foci, and TT' will thus be the minor axis of the desired unique conic.

While, with C falling anywhere outside IJ along TT', then the perpendicular through it will now be the minor axis of the required unique conic, whose major is TT'. And the fixed ratio $CI \cdot CJ = CS^2 = CS'^2$ gives us the desired foci.

Furthermore, we can point out the limits of C, with regard to any desired conic, as follows.

Join $E'E$ and HH', cutting, and being perpendicularly bisected by TT' in points v and w, respectively. Let u be the medial point of magnitude TT', and let het, $h'e't'$ be the symmetric and congruent medial triangles of HET and $H'E'T'$, respectively.

Then, obviously, he and $h'e'$ bisect in common TT' in u; and, in like manner, hvt and $h'vt'$ concur in v, and etw and $e't'w$ concur in w, and not otherwise. Were the symmetric medial triangles of $E'T'H$ and $ET'H'$ taken, their produced sides must likewise concur in the said three points u, v, w, along TT'.

Lastly, let K be the center of the circle through points I, E', H, J, H', E. We will show this to be also the common intersection of TT' by the circumcircles of ETH, $E'TH'$, $E'T'H$, and $ET'H'$.

(a) Then, with the given center C at either I or J, there evidently results either the common incircle or excircle of ETH and $E'TH'$, or one of the two common excircles of $E'T'H$ and $ET'H'$, as the case may be.

(b) With C at either v or w we have, in each case, a degenerated right line conic, having foci at E', E or H, H', as the case may be; each of which may be considered as the extreme form of, or limit between, an ellipse and a hyperbola.

(c) The center C falling anywhere between I and v gives us, as shown, foci falling symmetrically upon the arc $E'IE$, so that we now have an ellipse, inscribed to ETH and $E'TH'$, or escribed to $E'T'H$ and $ET'H'$, with TT' as its minor axis; while, in like manner, C falling anywhere between w and J gives an ellipse, with TT' as its minor axis, and foci which fall symmetrically upon arc HJH', escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$.

(d) And similarly, C falling anywhere between v, w gives us a hyperbola, whose minor axis is TT', escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having foci upon the symmetric arcs $E'H$ and EH', and the asymptotal angles which rise from zero, as C moves from v to T', attain the maximum value of $E'T'H$ when C is at T', where $ET'H$ and $E'T'H$ are the asymptotes; and sink again towards zero as C moves from T' to w; since, by the law of the hyperbola, real tangents concurring upon the minor axis, and thus touching opposite branches of the curve, must ever concur in an angle less than the supplementary asymptotal angle (this being their ideal maximum), and hence must concur supplementarily in an angle ever greater than the said asymptotal angle.

(e) Next, taking C at u, the medial point of TT', we plainly obtain, once more, a degenerated ellipse or hyperbola, a right line conic, but now one coincident with TT', its major axis, and having foci at T, T'; a fact also evident from the consideration that $IT':IT=JT':JT$, or $uI \cdot uJ=uT^2=uT'^2$.

(f) From this it follows that if C be taken anywhere within the limits u and I, then the foci S, S' must fall, one between T, I and the other between T', I, to fulfill our ratio $CI \cdot CJ=CS^2=CS'^2$, so that we now obtain an ellipse, inscribed to ETH and $E'TH'$, or escribed to $E'T'H$ and $ET'H$, and having TT' for its major axis.

(g) While, in like manner, C falling anywhere along TT', produced beyond J, gives us an ellipse, escribed in common to ETH and $E'TH'$, or to

$E'T'H$ and $ET'H'$, having TT' for its major axis, and foci which fall between J, K [as we will later prove], and upon TT' produced beyond J respectively.

(h) On the other hand, if C be taken along TT' produced, anywhere beyond u, the same ratio $CI\cdot CJ=CS^2=CS'^2$ necessitates that its two foci S, S' must fall outside the limits T, T', along TT' produced in both directions, so that now we obtain a hyperbola, escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having TT' for its major axis; and an asymptotal angle, which rises from zero, as C moves from u towards T, attains its maximum value of HTH' when C is at T, and HT, $H'T$ are the asymptotes; and sinks again towards zero as C retreats from T along TT' produced towards infinity. For obviously, as before stated, by the law of the hyperbola, the concurrent angle HTH' of the tangents TH and TH', which touch the same branch of the curve, must ever be greater than their ideal asymptotal angle.

(i) Lastly, escribing a parabola in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having TT' for its major axis, then, in either case, its focus S, by a well known theorem, must lie on the circumcircles of the said circumscribing triangles ETH and $E'TH'$, or $E'T'H$ and $ET'H'$; while by the foregoing Corollary 2, this focus S must also be the center of the circle through $E'HEH'$, and hence K, this center, is also the common intersection of axis TT', as stated, by the four circumcircles of ETH, $E'TH'$, $E'T'H$ and $ET'H'$. And thus, when the center of our conic has retreated to infinity in either direction along TT', then K will be the position of the focus of the resultant parabola, escribed to ETH and $E'TH'$, or to $E'TH$ and $ET'H'$.

(h) So that, returning to the previously mentioned hyperbola, whose center is beyond u, along TT' produced, and whose major axis is TT', it further follows that one of its foci must ever fall between the limits T', K, and thus within the circumcircles of its circumscribing triangles ETH and $E'TH'$, or $E'T'H$ and $ET'H'$; and similarly, as stated, one focus of the ellipse, escribed to the same triangles, must fall within the limits K, J and outside those circumcircles. Results independently proved in a previous theorem of mine (*Supplemento ai Rendiconti del Circolo Matematico di Palermo*).

Corollary 5. Wherefore, were we given any two tangents to an unspecified conic, Et and Et', or ET and ET', concurring in E, and were also given the position of the said conic's two axes, and thus its center, but knew neither the magnitude, nor the names of the said axes, then in every case (with the exceptions yet to be noted, in Corollary 6) a unique solution is given by the foregoing theorem, that is, one conic and one only, fulfilling the required conditions of touching the two given tangents, and having its axes in the given positions.

For, letting Et and Et' (Fig. 1) or ET and ET' (Fig. 2) be our two given tangents, with tCt' and ACA', or TCT' and BCB', as the two unspecified axes, we can, obviously, take either of these two axes as the axis of symmetry for our second pair of tangents $E't$ and $E't'$, or $E'T$ and $E'T'$, cutting our first pair in t, g and g', t', or in T, g and g', T', respectively; and giving us then two congruent and symmetric triangles Etg and $E'tg'$, or ETg and $E'Tg'$, whose vertical angles at t or T are externally bisected by the said axis of symmetry tt' or TT'. From which it follows, as shown, that the circle through $EgIg'E'J$ either cuts the other axis ACA' in S, S', the foci of the required conic; or else, if it cuts that other axis merely in imaginary points, then the ratio $CI\cdot CJ=CS^2=CS'^2$ gives us these foci.

And note: if the said circle through $EgIg'E'J$ has its center on the conic's major axis, then C is external to IJ; while if its center is on the minor, then C is within IJ; and hence, were we to draw EI, or EJ, the internal or external bisector of the concurrent angle tEt' or TET', according to whether C is nearer to I or to J, then the said bisector EI or EJ will ever cut and cross the major axis of our conic before it can cut the minor in I or J, as the case may be.

So that, conversely, with the given concurrent tangents Et and Et', or ET and ET', and the two unspecified axes, if the concurrent angle tEt' or TET' be bisected, internally, or externally, by EI or EJ, as the case may be, then this bisector will ever cut and cross first that axis which is to be the major of our unknown conic, an important fact to know.

But, if the said bisector should happen to pass through C, and cut neither axis first, then the conic must thereby be the incircle or excircle of tEt' or TET', according to whether the said bisector is the internal or external bisector of tEt' or TET'.

Lastly: with C as a known point, the desired conic is plainly central. But were one of the two axes and C specified as lying at infinity, then we must evidently have a parabola, and the medial point K of the points I and J wherein our internal and external bisectors of TET' cut our known axis TT', will be its focus, and the parabola can thus be drawn.

Corollary 6. But, as in Corollary 4, an exception to the foregoing evidently occurs in the special case of the concurrent point of our given two tangents falling upon one of our given axes.

For if the said axis does not bisect the concurrent angle, then we merely obtain, by axial symmetry, two sets of concurrent rays, and no conic is possible, other than in the degenerated form of the said axis itself.

While, with the axis bisecting internally or externally the concurrent angle, then the twofold axial symmetry gives us a rhombus[*] within which we can plainly place an incircle and innumerable inscribed ellipses, each having either of the two axes for its major, while either axis can also be the major axis of innumerable escribed hyperbolas, whose asymptotal angles vary from zero, up to an ideal maximum, that they can never reach, equal to that of the rhombic angles which lie on the chosen major axis.

Corollary 7. In the theorem, the circle whose diameter is the "limiting point," and thus the inter-focal distance of the real or imaginary foci, as the case may be, plainly cuts all the co-axial circles orthogonally.

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[*] So that our conic is here indeterminate. And further data are needed.

Corollary 8. In the theorem, the hyperbola's asymptotes being also tangents, the circle through the points (say e, f, e', f') in which any two tangents cut them, is also one of the co-axial system having the minor axis for its radical axis. So that $Ce.Cf=CS^2=Ce'.Cf'$.

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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

NOTE ON SHORT METHODS IN ARITHMETICAL CALCULATIONS.

From time to time, we have sent to us for publication, "short cuts" and "lightning methods." Most of these are of no theoretical and little practical value, and hence cannot be given a place in the MONTHLY.

The short cuts below were sent to us by Mr. Charles H. Case of Chicago, who is now neariy 81 years old. He says the list, consisting of eighteen examples, was prepared for the students of Wheaton College, November, 1896. We publish a few of them because we have found some of them useful in practical computation, having used them for years.

Mr. Case says, "The examples given should be wrought without the use of more figures than are used in the same." The principles used may be found mainly in the algebraic formulae given below.

$$(a\div b)(b\div a)=1;\quad (a+b)^2=a^2+2ab+b^2;\quad (a+b)(a-b)=a^2-b^2).$$

1. $(5\frac{1}{2})^2=30\frac{1}{4}$; $(7\frac{1}{2})^2=56\frac{1}{4}$; $(65)^2=4225$; $(88)^2=7744$; $96^2=9216$; $36^2=1296$; $76^2=5776$.

2. $625^2=\begin{array}{r}360625\\ \underline{3}\\ 390625\end{array}$; $876^2=767376$; $2496^2=6230016$.

3. $99649964125^2=\begin{array}{r}9928129699281296015625\\ +198562592\\ +249124910\\ \hline 9930115350113787015625\end{array}$

4. $(2436)^2=\begin{array}{r}5761296\\ \underline{1728}\\ 5934096\end{array}$; $68\times132=8976$; $8852\times8948=79207696$; $868\times932=808976$.